

Use of dual numbers in kinematical analysis of spatial mechanisms. Part II: applying the method for the generalised Cardan mechanism

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Abstract. The equations which permit obtaining the displacements from kinematical pairs were deduced in the first part of the paper. An actual case is presented in the second part of the work, the method being applied for the generalized Cardan mechanism. In a first stage, the parameters required for the completion of the dual matrices from the dual matrix equation of closing the kinematical chain are identified. Next, the matrix equation which allow for finding the rotations and the translations, respectively, from the pairs, are identified. A special attention is assumed on establishing the rotations from the pairs. The dual matrix equation is split into two real matrix equations, corresponding to rotations and to the translational motions, respectively. The equation corresponding to the translations presents a complex form, but it is simplified by using some characteristics of the matrices. Additionally, this matrix equation is written under a form which allows for algorithmization. Finally, there are obtained the equations which describe the position of the mechanism - the same as in the classical approach.

1. Introduction

The second part of the paper presents an application of the theoretical results deduced in the first part of the work. In figure 1 is presented the image of an actual Cardan joint. The Cardan joint is a spherical mechanism used for transmitting the rotational motion between concurrent axes. It consist in a driving element 1, an intermediate element (Cardanic cross) 2, a driven element 3 and an immobile element 4, the ground. All the joints of the mechanism are rotational pairs. The Hartenberg-Denavit [1] method requires denoting the axes of the four pairs by $z_k, k = 1..4$ and then defining the common normals of the couples of axes from the same element. To be mentioned that for spherical mechanisms the length of the common normal for any two axes is null and thus it will be defined as the straight line normal to the plane formed by the two axes, passing through the intersection point. Additionally, in the matrix closure equation, all the vectors corresponding to translations must be identical zero. For the Cardan mechanism from figure 1, the matrix closure equation is written as:

$$z(\theta_1)x(\alpha_{12})z(\theta_2)x(\alpha_{23})z(\theta_3)x(\alpha_{34})z(\theta_4)x(\alpha_{41})=I_3 \quad (1)$$

In equation (1), the position angle of the driving element is θ_1 and the angles θ_2 , θ_3 and θ_4 corresponding to the motions from the other pairs must be found. The arguments of the x matrices represent the constant constructive parameters, with the values:

$$\alpha_{12} = \pi / 2 ; \alpha_{23} = \pi / 2 ; \alpha_{34} = \pi / 2 ; \alpha_{41} = \pi - \alpha \quad (2)$$

In the relation (2), α is the acute angle between the driven and driving axes. When the rotational motion ought to be transmitted between two crossed axes, a versatile solution consists in constructing a mechanism similar to the Cardanic transmission.

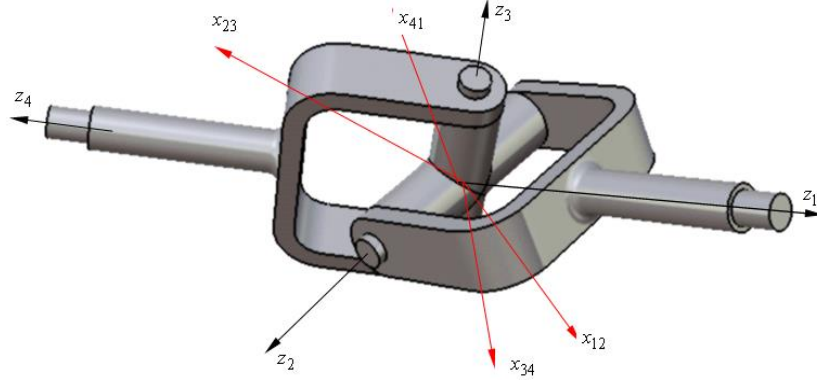


Figure 1. The Cardan mechanism

The family of the mechanism with crossed axes has the value 1, the mobility of the mechanism is:

$$M_0 = 6 \cdot n - 4c_5 = 6 \cdot 3 - 4 \cdot 5 = -2 \quad (3)$$

and therefore the Cardanic transmission cannot be directly used because the mechanism is blocked.

2. Use of dual matrices for obtaining the closure equation of the generalised Cardan mechanism

According to the transfer principle due to Kotelnikov [2], all the relations from spherical trigonometry are valid when the real angles are replaced by dual angles. From this replacement, it results that all rotational motions from the spherical mechanisms become helicoidally motions. Another consequence is the transformation of the spherical mechanisms into spatial ones. This remark led to the idea of studying the kinematics of the generalized Cardan mechanism based on the kinematics of Cardan transmission. The simplest way to “relax” the mechanism is by transforming three of the rotational pairs into cylindrical pairs, the mechanism becoming a RCCC one. The kinematical analysis of the RCCC mechanism for random values of the constructive parameters was made by Yang [3], using the method of dual quaternions. The test rig presented in figure 2 is considered for applying the method.

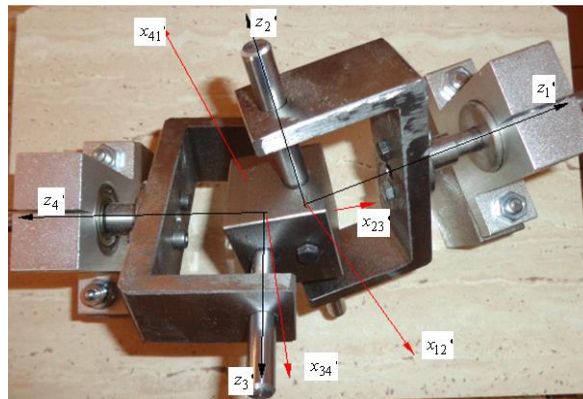


Figure 2. Generalized Cardan mechanism

The closure equation for the contour is obtained by transforming the angles from the equation (1) into dual angles, according the equations deduced in the first part of the work [4]. The closure equation is:

$$\mathbf{z}(\hat{\theta}_1) \mathbf{x}(\hat{\alpha}_{12}) \mathbf{z}(\hat{\theta}_2) \mathbf{x}(\hat{\alpha}_{23}) \mathbf{z}(\hat{\theta}_3) \mathbf{x}[\hat{\alpha}_{34}] \mathbf{z}(\hat{\theta}_4) \mathbf{x}(\hat{\alpha}_{41}) = \hat{\mathbf{I}}_3 \quad (4)$$

The form:

$$\begin{aligned} & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & \mathcal{E} \left[\begin{aligned} & \mathbf{z}(\theta_1)(s_1 \mathbf{Q}_\theta) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12})(a_{12} \mathbf{Q}_\alpha) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2)(s_2 \mathbf{Q}_\theta) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23})(a_{23} \mathbf{Q}_\alpha) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3)(s_3 \mathbf{Q}_\theta) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34})(a_{34} \mathbf{Q}_\alpha) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4)(s_4 \mathbf{Q}_\theta) \mathbf{x}(\alpha_{41}) + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41})(a_{41} \mathbf{Q}_\alpha) \end{aligned} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mathcal{E} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5) \end{aligned}$$

is obtained as it follows: the left member of the equation (4) is expanded and contains the real part and the dual part; the dual part is identical to the left member of the equation of closure (1) the spherical mechanism; the dual part is a sum with the number of terms equal to the number of factors of the real part; each of these terms is obtained by introducing in each term of the dual part the matrix $s\mathbf{Q}_\theta$ or $a\mathbf{Q}_\alpha$, depending on the type of the preceding matrix, $\mathbf{z}(\theta)$ or $\mathbf{x}(\alpha)$. The equation was written under the assumption that all the pairs of the mechanism were transformed into cylindrical pairs, in order to present a higher degree of generality for the solution. The equation (5) is equivalent to two equations obtained by equating the real and dual parts, respectively, from the members of equation (5), from which one is the equation (1) itself. The second equation seems complicated at first sight but it can be simplified based on equation (1). Following the remark that any term of the dual part is split by the $\mathbf{Q}_\alpha$ or $\mathbf{Q}_\theta$ matrix into two factors, any of these factor is the transpose of the other one, and the next equation is obtained:

$$\begin{aligned} & \mathbf{z}(\theta_1)(s_1 \mathbf{Q}_\theta) [\mathbf{z}(\theta_1)]^T + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12})(a_{12} \mathbf{Q}_\alpha) [\mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12})]^T + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2)(s_2 \mathbf{Q}_\theta) [\mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2)]^T + \\ & \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23})(a_{23} \mathbf{Q}_\alpha) [\mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23})]^T + \\ & [\mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41})]^T (s_3 \mathbf{Q}_\theta) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & [\mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41})]^T (a_{34} \mathbf{Q}_\alpha) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\ & [\mathbf{x}(\alpha_{41})]^T (s_4 \mathbf{Q}_\theta) \mathbf{x}(\alpha_{41}) + \\ & \mathbf{I}_3 (a_{41} \mathbf{Q}_\alpha) = \mathbf{O}_3 \quad (6) \end{aligned}$$

Considering the equations obtained in the part I of the paper [4]:

$$\mathbf{z}(\theta) \mathbf{Q}_\theta \mathbf{z}(\theta)^T = \mathbf{Q}_\theta; \quad \mathbf{x}(\alpha) \mathbf{Q}_\alpha \mathbf{x}(\alpha)^T = \mathbf{Q}_\alpha \quad (7)$$

The equation (6) can now be simplified:

$$\begin{aligned}
& s_1 \mathbf{Q}_\theta + \\
& \mathbf{z}(\theta_1)(a_{12} \mathbf{Q}_\alpha) [\mathbf{z}(\theta_1)]^T + \\
& \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12})(s_2 \mathbf{Q}_\theta) [\mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12})]^T + \\
& \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2)(a_{23} \mathbf{Q}_\alpha) [\mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2)]^T + \\
& [\mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41})]^T (s_3 \mathbf{Q}_\theta) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\
& [\mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41})]^T (a_{34} \mathbf{Q}_\alpha) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) + \\
& [\mathbf{x}(\alpha_{41})]^T (s_4 \mathbf{Q}_\theta) \mathbf{x}(\alpha_{41}) + \\
& a_{41} \mathbf{Q}_\alpha = \mathbf{O}_3
\end{aligned} \tag{8}$$

To be underlined that for the calculus of the dual part of the matrix closure equation (5) there are necessary $n = 64$ matrix products while for the equivalent form, relation (8), only $n = 2(2 + 4 + 6) = 24$ matrix products are required, meaning 37.5% from the initial number of products, thus considerably reducing the time for calculus. As shown in the part I of the paper [4], the matrices resulting from products of $\mathbf{MQ}_\alpha \mathbf{M}^T$, $\mathbf{MQ}_\theta \mathbf{M}^T$ type are antisymmetric, and it results that the left member of equation (8) is a sum of antisymmetric matrices and it will be an antisymmetric matrix, too.

In conclusion, the equation (8) will provide three independent scalar equations. The angles of rotation from cylindrical pairs are present in these equations and thus the equation (1) must first be solved in order to obtain these angles and after that the translations from the pairs of the mechanism will be found.

3. The solutions of dual matrix closure equation. Observations and discussions

For the present case, three from the six unknowns are rotations (and three are translations) which can be found using only the matrix equation (1). But, as McCarthy shows [5], complicated calculus occurs when using the matrix equation under the form (1). To avoid this aspect, the equality is re-written in a convenient form, as the equality between two matrices. For the actual case, the matrix equation (1) is re-written as:

$$\mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) = \mathbf{I}_3 \mathbf{z}(\theta_1) \tag{9}$$

$$[\mathbf{z}(\theta_1)]^T \mathbf{z}(\theta_1) \mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) \mathbf{z}(\theta_1) = \mathbf{z}(\theta_1) \tag{10}$$

and it results:

$$\mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) \mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) \mathbf{z}(\theta_1) = \mathbf{I}_3 \tag{11}$$

The equation (11) is expressed now as:

$$\mathbf{x}(\alpha_{12}) \mathbf{z}(\theta_2) \mathbf{x}(\alpha_{23}) \mathbf{z}(\theta_3) = [\mathbf{x}(\alpha_{34}) \mathbf{z}(\theta_4) \mathbf{x}(\alpha_{41}) \mathbf{z}(\theta_1)]^T \tag{12}$$

The particular values $\alpha_{12} = \pi / 2$, $\alpha_{23} = \pi / 2$ and $\alpha_{34} = \pi / 2$ lead to the explicit form:

$$\begin{aligned}
& \begin{bmatrix} \cos \theta_2 \cos \theta_3 & -\cos \theta_2 \sin \theta_3 & \sin \theta_2 \\ -\sin \theta_3 & -\cos \theta_3 & 0 \\ \sin \theta_2 \cos \theta_3 & -\sin \theta_2 \sin \theta_3 & -\cos \theta_2 \end{bmatrix} = \\
& = \begin{bmatrix} \cos \theta_1 \cos \theta_4 - \cos \alpha_{41} \sin \theta_1 \sin \theta_4 & -\sin \theta_{41} \sin \theta_1 & \cos \theta_1 \sin \theta_4 + \cos \alpha_{41} \sin \theta_1 \cos \theta_4 \\ -\sin \theta_1 \sin \theta_4 - \cos \theta_1 \cos \alpha_{41} \sin \theta_4 & -\sin \theta_{41} \cos \theta_1 & -\cos \theta_1 \sin \theta_4 + \cos \alpha_{41} \cos \theta_1 \cos \theta_4 \\ \sin \alpha_{41} \sin \theta_4 & -\cos \alpha_{41} & -\sin \alpha_{41} \cos \theta_4 \end{bmatrix}
\end{aligned} \tag{13}$$

The matrix equation (13) allows for finding the rotations from the three cylindrical pairs of the mechanism. Details upon the methodology are exhaustively given in [6], using special inverse trigonometrically functions [7-8].

4. Conclusions

The paper presents an example of applying the method of dual numbers in the kinematical analysis of spatial mechanisms. Initially, the Cardan mechanism that is a spherical mechanism, is considered and the closure matrix equation is written using rotation matrices of $(3,3)$ type with real elements.

For the spatial mechanism obtained from the Cardan mechanism after the rotation pairs were transformed into cylindrical pairs, the closure matrix equation is obtained, according to Kotelnikov's principle, by the plain transformation of the actual angles from the closure equation of the spherical mechanism, into dual angles.

The dual equation is written according to the results obtained in Part I of the paper and leads to two matrix equations, resulting after equating the real parts and imaginary parts of the closing matrix equation. The first equation allows for finding the rotations from the pairs of the mechanism and the second equation helps finding the translations from the pairs. Some remarks concerning the method of finding the rotations from the pairs with the avoidance of strange solutions are made.

5. References

- [1] Hartenberg R S and Denavit J 1965 *Kinematic synthesis of linkages* (McGraw-Hill series in mechanical engineering, New York McGraw-Hill)
- [2] Kotelnikov A P 1950 Theory of Vectors and Complex Numbers, in *Several Applications of Lobachevskii's Ideas in Mechanics and Physics* (Collected Papers of Gostekhizdat Moscow) 7-47
- [3] Yang A T and Freudenstein F 1964 Application of dual number quaternions algebra to the analysis of spatial mechanisms *ASME J. Eng. Ind.* **86** 300–308
- [4] Ciornei F C, Alaci S, Pentiuc R D and Doroftei I 2019 Use of dual numbers in kinematical analysis of spatial mechanisms. Part I: principle of the method *IMT Oradea Ann. Ses. Scient. Papers*
- [5] McCarthy J M 2000 *Geometric Design of Linkages* (Springer Berlin)
- [6] Ciornei F C, Alaci S, Romanu I C, Mihai I and Tibu G A 2019 Kinematical analysis of a generalized Cardanic joint *IOP Conf. Ser.: Mater. Sci. Eng.* 477 012037
- [7] Hahn B and Valentine D 2019 *Essential MATLAB for Engineers and Scientists* 7th Edition, (Academic Press)
- [8] Maxfield B 2009 *Essential Mathcad for Engineering* Second Edition (Academic Press)

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